

# Algebraic treatment of a simple model for the electromagnetic self-force

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**Abstract** The problem of the electromagnetic self-force can be studied in terms of a quadratic PT-symmetric Hamiltonian. Here, we apply a straightforward algebraic method to determine the regions of model-parameter space where the quantum-mechanical operator exhibits real spectrum. An alternative point of view consists of finding the values of the model parameters so that a symmetric operator supports bound states.

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## 1 Introduction

In a recent paper Bender and Gianfreda [1] discussed a model for the electromagnetic self-force on an oscillating charge particle. They showed that the problems reported by Englert [2] were due to the fact that the PT-symmetric Hamiltonian that describes the system exhibits broken PT symmetry. The addition of two interacting terms to that Hamiltonian enabled Bender and Gianfreda to analyse the regions of broken and unbroken PT symmetry. To this end the authors resorted to the approach of Rossignoli and Kowalski [3] in order to rewrite the quadratic Hamiltonian in diagonal form. The authors mentioned the possibility of using an alternative approach for the same purpose [4] which is known since long ago [5].

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The purpose of this paper is to stress the fact that the application of the latter algebraic method to quadratic Hamiltonians is extremely simple and straightforward. In section 2 we derive the regular or adjoint matrix representation of the modified Hamiltonian operator for the electromagnetic self-force [1] and obtain the spectral frequencies in terms of the model parameters. These frequencies, which are roots of the characteristic polynomial for the matrix already mentioned, are suitable for determining the regions in parameter space where the PT symmetry is broken. Finally, in section 3 we summarize the main results and draw conclusions.

## 2 The quantum-mechanical model

From the pair of classical equations of motion proposed by Englert [2], Bender and Gianfreda [1] derived the Hamiltonian function

$$H_c = \frac{p_x p_w - p_y p_z}{m\tau} + \frac{2p_z p_w}{m\tau^2} + \frac{wp_y + zp_x}{2} - \frac{mzw}{2} + kxy, \quad (1)$$

where  $q = x, y, z, w$  are suitable coordinates and  $p_q$  the corresponding conjugate momenta. The quantum-mechanical version of this operator is PT-symmetric but its eigenvalues are not real because the PT symmetry is broken for all  $m, \tau, k$ . To overcome this difficulty Bender and Gianfreda added two terms and obtained the modified Hamiltonian operator

$$H = \frac{B(wp_z - zp_w)}{m\tau} + \frac{2p_z p_w}{m\tau^2} + \frac{p_x p_w - p_y p_z}{m\tau} - \frac{mzw}{2} + \frac{wp_y + zp_x}{2} + kxy + \frac{A(x^2 + y^2)}{2}, \quad (2)$$

where the coordinates and momenta satisfy the quantum-mechanical commutation relations  $[q, p_q] = i$ . In this way those authors were able to show that the resulting PT-symmetric Hamiltonian exhibits real spectrum for suitable values of  $A$  and  $B$ . In particular, the PT symmetry is broken for the case  $A = B = 0$  that leads to  $H_c$ .

The simplest way of determining the regions of broken PT symmetry is to convert  $H$  into a suitable diagonal form. Bender and Gianfreda resorted to the approach proposed by Rosignoli and Kowalski [3] and in what follows we apply a well known algebraic approach[F96] proposed recently by Fernández [4] for the treatment of a simple model for optical resonators [6].

The algebraic approach is based on the construction of the adjoint or regular matrix representation of  $H$  in the operator basis  $\{O_1, O_2, \dots, O_8\} = \{x, y, z, w, p_x, p_y, p_z, p_w\}$ . The matrix elements  $H_{ij}$  are given by the coefficients of the commutator relations

$$[H, O_i] = \sum_{j=1}^8 H_{ji} O_j, \quad i = 1, 2, \dots, 8. \quad (3)$$

A straightforward calculation leads to

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & iA & ik & 0 & 0 \\ 0 & 0 & 0 & 0 & ik & iA & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & \frac{iB}{m\tau} & 0 & 0 & 0 & -\frac{im}{2} \\ 0 & -\frac{i}{2} & -\frac{iB}{m\tau} & 0 & 0 & 0 & -\frac{im}{2} & 0 \\ 0 & 0 & 0 & -\frac{i}{m\tau} & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & \frac{i}{m\tau} & 0 & 0 & 0 & 0 & \frac{i}{2} \\ 0 & \frac{i}{m\tau} & 0 & -\frac{2i}{m\tau^2} & 0 & 0 & 0 & \frac{iB}{m\tau} \\ -\frac{i}{m\tau} & 0 & -\frac{2i}{m\tau^2} & 0 & 0 & 0 & -\frac{iB}{m\tau} & 0 \end{pmatrix}. \quad (4)$$

The spectral frequencies of the problem are the eigenvalues of this matrix. The secular equation  $|\mathbf{H} - \lambda \mathbf{I}| = 0$  (where  $\mathbf{I}$  is the  $8 \times 8$  identity matrix) yields the characteristic polynomial

$$(m^2 \tau^2 \xi - B^2 + m^2) (m^2 \tau^2 \xi^3 + \xi^2 (m^2 - B^2) + \xi (2AB - 2km) - A^2 + k^2) = 0, \quad (5)$$

where  $\xi = \lambda^2$ . One of the roots is

$$\xi = \frac{B^2 - m^2}{m^2 \tau^2}, \quad (6)$$

and the remaining three ones are solutions to the cubic equation

$$m^2 \tau^2 \xi^3 + (m^2 - B^2) \xi^2 + 2(AB - km) \xi - A^2 + k^2 = 0. \quad (7)$$

When the spectrum is real the frequencies are real and the four roots  $\xi_j$  are positive. Thus the region in parameter space where the spectrum is real is determined by the set of equations  $\{B^2 > m^2, A^2 > k^2, AB > km\}$ .

Bender and Gianfreda [1] obtained exactly the same equations (6) and (7) by means of an alternative approach based on the creation and annihilation operators

[3]. In our opinion the adjoint or regular matrix representation of the Hamiltonian operator provides a more straightforward route. In addition to it, the algebraic approach is somewhat more general because it applies to any Hamiltonian that satisfies the commutator relations (3) [5].

### 3 Conclusions

Any Hamiltonian that is a quadratic function of the coordinates and momenta or of the creation and annihilation operators is exactly solvable. One way of obtaining its spectrum is to convert it into a diagonal form by means of a linear combination of the relevant dynamical variables [3]. The algebraic method [4, 5] is a particularly simple approach because the construction of the adjoint or regular matrix representation is straightforward. It only requires the calculation of simple commutation relations like (3).

It is also worth mentioning that the quantum-mechanical versions of  $H_c$  and  $H$  are symmetric operators. They satisfy  $\langle f | H | g \rangle = \langle g | H | f \rangle^*$  for any pair of square integrable functions  $f(\mathbf{q})$  and  $g(\mathbf{q})$ . If  $\psi$  is a normalizable eigenfunction of  $H$  ( $H\psi = E\psi$ ,  $\langle \psi | \psi \rangle < \infty$ ) then  $E$  is real. Therefore, in the present case, looking for the regions in parameter space of unbroken PT symmetry is equivalent to looking for the regions where  $H$  supports bound states (as Bender and Gianfreda [1] did explicitly for the ground state). The same situation emerged in the case of the optical resonators mentioned above [4, 6].

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